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Application of the method of moments for calculating the dynamic response of periodically driven nonlinear stochastic systems

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Abstract

It is shown that the method of moments allows one to calculate the first- and higher-harmonic susceptibilities of nonlinear stochastic systems with high accuracy. The dependence of the spectral amplification at the first three harmonics on the noise intensity is studied. It is shown that stochastic resonance at the third harmonic occurs at two separate values of the noise intensity for not too large a bias. Also, it is demonstrated that even when the bias is so large that the bistable system turns into a monostable one, the stochastic resonant enhancement of the system's sensitivity to the external driving field is still observed at some optimal non-zero value of the noise intensity.

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1. Introduction

The dynamics of nonlinear periodically driven stochastic systems has been the subject of intense investigations. The primary interest in these systems is due to the fact that they exhibit a phenomenon of stochastic resonance (SR). SR manifests itself in a non-monotonic dependence of the signal-to-noise ratio and spectral power amplification on the noise intensity, D; namely, at a given driving frequency, these two quantities exhibit a peak at some optimal value of D. The phenomenon of SR was originally proposed as being responsible for the periodic recurrence of the ice ages of the Earth [1], and later was observed experimentally in such systems as the Schmitt trigger [2], magnetic submicron particles [3], analogue circuits [4], ring lasers [5], etc.

At the theoretical level, exact expressions describing the amplification of external signals in nonlinear stochastic systems can be obtained only for some special cases; for example, SR with multiplicative noise [6] or in a special type of potential such as a bistable piecewise linear potential [7, 8]. Therefore, various approximate techniques have been introduced to calculate the linear and nonlinear susceptibilities of stochastic systems. Of interest are the Langevin-equation-based analysis [3, 16], adiabatic approximation [17], the two-state theory [9], linear response theory [10], perturbation theory [11] and the matrix continued fraction method [12–15, 18]. However, the above-mentioned methods are characterized by long computation times when high accuracy is required, are not easily programmed or their application is restricted to a narrow parameter range.

The purpose of this paper is to introduce an efficient technique to study the linear and nonlinear susceptibilities of driven stochastic systems. The proposed technique is based on the method of moments. Although this method has been used previously [16] to calculate the linear amplification of the external signal, to the best of our knowledge, it has not been applied directly to studying the amplification properties at higher harmonics.

This paper is organized as follows. In the next section, the application of the method of moments to the problem of computing the linear and nonlinear susceptibilities of a stochastic system in a consistent manner is described. Then, its accuracy is demonstrated by a comparative analysis of the results obtained using the proposed method, numerical solution of the Fokker–Planck equation and high-frequency asymptotic formulae derived in section 3. In section 4, the method of moments is applied to study the noise intensity dependence of linear and nonlinear spectral amplification factors in both bistable and monostable systems. Finally, the results of the calculations are discussed and the conclusions are presented.

2. Susceptibility calculation using the method of moments

Let us consider the dynamic response of an ensemble of biased overdamped oscillators characterized by a coordinate x and Hamiltonian

$$H(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - hx \tag{1}$$

where h is an external bias. The oscillators are driven by Gaussian delta-correlated white noise of intensity D and external field $\varepsilon(t)$. The Fokker–Planck equation (FPE) for the probability density $W(x,t)$ is

$$\frac{\partial W(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial W}{\partial x} + \frac{\mathrm{d}H}{\mathrm{d}x}W - \varepsilon(t)W\right). \tag{2}$$

Let us assume that the external field $\varepsilon(t) = \varepsilon_0 \mathrm{e}^{\mathrm{i}\Omega t}$ is harmonic in time with angular frequency Ω and is so weak that the linear response theory is applicable to our system.

Multiplication of the FPE (2) by some function of x, $A(x)$, and integration over x from $-\infty$ to ∞ yields the following equation of motion for $\langle A\rangle$ (provided that the corresponding averages exist):

$$\frac{\mathrm{d}\,\langle A\rangle}{\mathrm{d}t} = D\left\langle\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right\rangle - \left\langle G\frac{\mathrm{d}A}{\mathrm{d}x}\right\rangle + \varepsilon(t)\left\langle\frac{\mathrm{d}A}{\mathrm{d}x}\right\rangle \tag{3}$$

where $G(x) = \mathrm{d}H/\mathrm{d}x$. Here, we have used integration by parts together with the fact that the distribution function and all of its derivatives vanish at infinity.

By Floquet's theorem, there exists a time-periodic solution of the FPE with the period of the external field, i.e.

$$W\left(x, t + \frac{2\pi}{\Omega}\right) = W(x,t). \tag{4}$$

This solution can be decomposed into the Fourier series

$$W(x,t) = \sum_{n=0}^{\infty} W_n(x)\,\varepsilon^n(t). \tag{5}$$

Conservation of the norm of the distribution function $W(x,t)$ implies that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, W_n(x) = 0 \qquad \text{for} \quad n \geqslant 1. \tag{6}$$

The term $W_0(x)$ represents the solution of the stationary FPE without external driving

$$W_0(x) = \frac{1}{Z}\mathrm{e}^{-H(x)/D} \tag{7}$$

where the partition integral is given by

$$Z = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-H(x)/D}. \tag{8}$$

The average $\langle A\rangle$ depends on time as

$$\langle A\rangle = \sum_{n=0}^{\infty} \langle A\rangle_n\, \varepsilon^n(t) \tag{9}$$

with

$$\langle A\rangle_n = \int_{-\infty}^{\infty} \mathrm{d}x\, A(x)\, W_n(x). \tag{10}$$

Substitution of the expansion (9) into the equation of motion for $\langle A\rangle$ (3) yields the following set of coupled equations for $\langle A\rangle_n$:

$$D\left\langle \frac{\mathrm{d}^2 A}{\mathrm{d}x^2}\right\rangle_0 - \left\langle G\frac{\mathrm{d}A}{\mathrm{d}x}\right\rangle_0 = 0 \tag{11}$$

$$D\left\langle \frac{\mathrm{d}^2 A}{\mathrm{d}x^2}\right\rangle_n - \left\langle G\frac{\mathrm{d}A}{\mathrm{d}x}\right\rangle_n + \left\langle \frac{\mathrm{d}A}{\mathrm{d}x}\right\rangle_{n-1} = \mathrm{i}n\Omega\, \langle A\rangle_n \qquad n \geqslant 1. \tag{12}$$

In deriving the last two equations, the orthogonality property of the functions $\mathrm{e}^{\mathrm{i}n\Omega t}$ was used.

We are looking for functions $W_n(x)$ in the form

$$W_n(x) = w_n(x) W_0(x). \tag{13}$$

In turn, the functions $w_n(x)$ are sought as power series

$$w_n(x) = \sum_{k=0}^{\infty} w_{n,k} x^k \tag{14}$$

with (see equation (7))

$$w_{0,k} = \delta_{0,k}. \tag{15}$$

Here $\delta_{i,j}$ is the usual Kronecker delta symbol. In order to find the unknown coefficients $w_{n,k}$, we require that equation (12) be satisfied for all moments, i.e. we take

$$A(x) = x^m \qquad m = 0, 1, 2, \ldots. \tag{16}$$

In the simplest case $m = 0$ (i.e. $A(x) = 1$), equation (12) yields

$$\sum_{k=0}^{\infty} w_{n,k} \left\langle x^k\right\rangle_0 = 0 \qquad n \geqslant 1 \tag{17}$$

in agreement with the norm conservation requirement (6). For other choices of $A(x)$, i.e. for $m \geqslant 1$, we obtain

$$\sum_{k=1}^{\infty}\left[Dmk\left\langle x^{k+m-2}\right\rangle_0 + \mathrm{i}n\Omega\left(\left\langle x^{k+m}\right\rangle_0 - \left\langle x^k\right\rangle_0\left\langle x^m\right\rangle_0\right)\right]w_{n,k} = m\sum_{k=0}^{\infty} w_{n-1,k}\left\langle x^{k+m-1}\right\rangle_0. \tag{18}$$

Here the property of norm conservation (17) was used. The nth-order susceptibility is

$$\chi^{(n)} \equiv \langle x\rangle_n = \sum_{k=0}^{\infty} w_{n,k}\left\langle x^{k+1}\right\rangle_0. \tag{19}$$

For the purposes of numerical calculation, one has to truncate the power series (14) at some value of k. In the following calculations, the upper value $k = 10$ was taken. It was found that in the parameter range considered, inclusion of higher powers of x did not change the results by more than 0.1%. The system (18) thus turned into a set of 10 linear equations that was solved numerically using Gaussian elimination procedure.

3. Accuracy analysis of the method of moments

With respect to the accuracy of the method of moments, it can be verified that at zero driving frequency, the method yields the analytically exact expressions for the static ($\Omega = 0$) susceptibilities, $\chi_{\mathrm{st}}^{(n)}$, namely

$$\begin{aligned}
\chi_{\mathrm{st}}^{(1)} &= \frac{1}{D}\left(\left\langle x^2\right\rangle_0 - \langle x\rangle_0^2\right)\\
\chi_{\mathrm{st}}^{(2)} &= \frac{1}{2D^2}\left(\left\langle x^3\right\rangle_0 - \langle x\rangle_0\left\langle x^2\right\rangle_0\right) - \frac{1}{D}\langle x\rangle_0\,\chi_{\mathrm{st}}^{(1)}\\
\chi_{\mathrm{st}}^{(3)} &= \frac{1}{6D^3}\left(\left\langle x^4\right\rangle_0 - \langle x\rangle_0\left\langle x^3\right\rangle_0\right) - \frac{1}{D}\langle x\rangle_0\,\chi_{\mathrm{st}}^{(2)} - \frac{1}{2D^2}\left\langle x^2\right\rangle_0\chi_{\mathrm{st}}^{(1)}.
\end{aligned} \tag{20}$$

These expressions are readily obtained after truncation of the power series (14) at $k = 1$ and are not affected by inclusion of higher powers of x in (14). As the driving frequency increases, the accuracy of the method becomes worse, and hence more terms must be included in the expansion (14). Therefore, in order to show that the method is accurate enough when only the first 10 terms are taken into account, we obtain approximate analytic expressions for the susceptibilities in the high-frequency limit, and also calculate the first-order susceptibility numerically from the FPE.

3.1. High-frequency approximation

In the following derivation, we generalize the approach adopted in [3] to calculate the signal-to-noise ratio at the second harmonic for a parabolic potential with a weak nonlinearity. We start with the Langevin equation of motion for the system's coordinate x

$$\dot{x} = -H'(x) + \varepsilon(t) + \Gamma(t) \tag{21}$$

where $\Gamma(t)$ is Gaussian delta-correlated white noise with zero mean; that is,

$$\langle\Gamma(t)\rangle = 0 \tag{22}$$

$$\left\langle\Gamma(t)\Gamma(t')\right\rangle = 2D\delta(t-t'). \tag{23}$$

In order to find the susceptibility of the system in the high-frequency limit, the coordinate x is decomposed into two parts:

$$x = x_r + x_f. \tag{24}$$

The component x_r describes the motion under the influence of the random force $\Gamma(t)$ only and satisfies the unperturbed Langevin equation

$$\dot{x}_r = -H'(x_r) + \Gamma(t). \tag{25}$$

This implies the Boltzmann distribution of x_r given by equations (7) and (8).

The component x_f corresponds to the oscillations due to the time-periodic external field $\varepsilon(t)$. Since the amplitude of the external field is small, one can treat x_f as a small parameter. Substituting the decomposition (24) into the original Langevin equation (21) and expanding the term $H'(x_r + x_f)$ into a Taylor series around x_r, the following equation of motion for x_f is obtained:

$$\dot{x}_f = -H''(x_r)x_f - \tfrac{1}{2}H'''(x_r)x_f^2 - \tfrac{1}{6}H^{(IV)}(x_r)x_f^3 + \mathrm{O}(x_f^4) + \varepsilon(t). \tag{26}$$

Here, equation (25) was used to eliminate the terms $\dot{x}_r$, $H'(x_r)$ and $\Gamma(t)$.

Now, we make an explicit assumption that the frequency of the external field is so large that the random component, x_r, does not change considerably during one period of the driving field. Then the terms involving various derivatives of $H(x_r)$ can be treated as constants and we can write

$$x_f(t) = \chi^{(1)}(x_r)\varepsilon(t) + \chi^{(2)}(x_r)\varepsilon^2(t) + \chi^{(3)}(x_r)\varepsilon^3(t) + \mathrm{O}(\varepsilon^4). \tag{27}$$

The proportionality coefficients $\chi^{(n)}(x_r)$ can be interpreted as being coordinate-dependent susceptibilities of the system. Substituting the expansion (27) into equation (26) and using the orthogonality of the functions $\mathrm{e}^{\mathrm{i}n\Omega t}$ yields the following expressions:

$$\begin{aligned}
\chi^{(1)}(x_r) &= \frac{1}{H''(x_r) + \mathrm{i}\Omega} \\
\chi^{(2)}(x_r) &= -\frac{1}{2}\frac{H'''(x_r)\left(\chi^{(1)}(x_r)\right)^2}{H''(x_r) + \mathrm{i}2\Omega} \\
\chi^{(3)}(x_r) &= -\frac{H'''(x_r)\chi^{(1)}(x_r)\chi^{(2)}(x_r) + \frac{1}{6}H^{(IV)}(x_r)\left(\chi^{(1)}(x_r)\right)^3}{H''(x_r) + \mathrm{i}3\Omega}.
\end{aligned} \tag{28}$$

The net nth-order susceptibility in the high-frequency limit is obtained after averaging $\chi^{(n)}(x_r)$ with respect to x_r:

$$\chi^{(n)} = \int_{-\infty}^{\infty} \mathrm{d}x_r\, \chi^{(n)}(x_r)\, W_0(x_r). \tag{29}$$

This completes the derivation.

We conclude this section with a note that the last expression suffers from what can be called an 'infrared catastrophe'. Considering the quartic potential (1), the absolute value of the susceptibility calculated from equations (28) and (29) unphysically increases as the driving frequency decreases, and diverges to infinity at zero driving frequency. However, equations (28) and (29) provide good approximate expressions for the dynamic susceptibility at large driving frequencies when the requirement of negligibly small change of x_r during one period of the applied field is approximately fulfilled.

3.2. Numerical calculation of susceptibility from the FPE

The first-order susceptibility was calculated using yet another method based on the numerical solution of the equation for the function $w_1(x)$ (cf equation (13)) derivable from the original FPE (2):

$$\mathrm{i}\omega w_1(x) = D\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} - \frac{\mathrm{d}H}{\mathrm{d}x}\frac{\mathrm{d}w_1}{\mathrm{d}x} + \frac{1}{D}\frac{\mathrm{d}H}{\mathrm{d}x} \tag{30}$$

with the conditions (see equation (6))

$$(W_0 w_1)|_{x\to\pm\infty} = 0 \tag{31}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, w_1(x)\, W_0(x) = 0. \tag{32}$$

In order to solve equation (30) numerically, a standard Runge–Kutta procedure was employed. Numerically exact values of the first-order susceptibility were found as

$$\chi^{(1)} = \int_{-\infty}^{\infty} \mathrm{d}x\, x\, w_1(x)\, W_0(x). \tag{33}$$

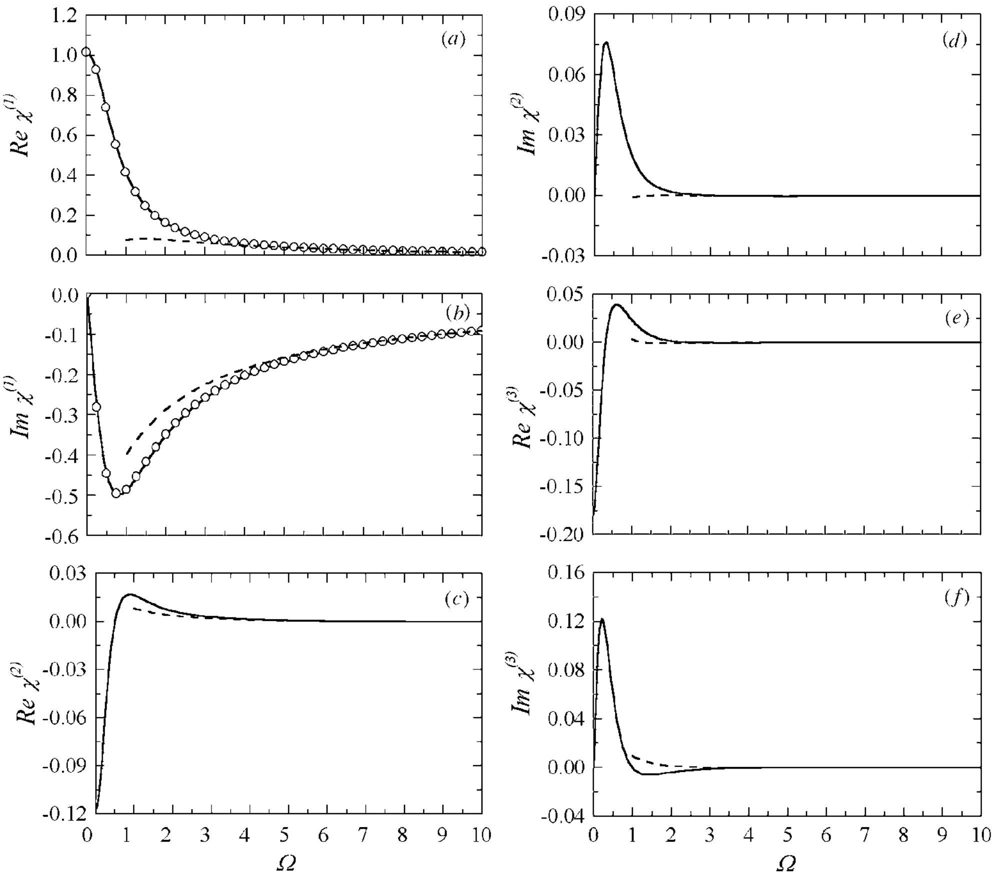

Figure 1. The dependence of the susceptibility on frequency at the first (*a*), (*b*), second (*c*), (*d*), and third (*e*), (*f*) harmonics. Plots (*a*), (*c*), (*e*) correspond to the real, and (*b*), (*d*), (*f*) to the imaginary part of susceptibility. The results are obtained from the method of moments (full curves), numerical solution of equation (30) (circles) and high-frequency approximation (broken curves).

3.3. Verification of the accuracy of the method of moments

In order to check the accuracy of the method of moments, the dependences of the first-, second- and third-order susceptibilities on frequency obtained from this method and from the high-frequency asymptotic formulae (28) and (29), as well as from numerical solution of equation (30), are compared in this section.

Figure 1 shows the dependence of the susceptibility on the driving frequency for the first three harmonics. The value of the external bias is $h = 0.2$ and the noise intensity is $D = 1$. The first-order susceptibility was calculated using the method of moments, the high-frequency asymptotic formula and numerical solution of equation (30); the higher-order susceptibilities were calculated using the first two methods.

A very good agreement between the results obtained from the method of moments (full curves) and numerically exact ones (open circles) is clearly seen from figures 1(*a*) and (*b*). The discrepancy between the two sets of data does not exceed the line thickness on the graph. It should be noted that the calculations based on the method of moments were about three orders of magnitude faster than those based on the numerical evaluation of the function w_1 from equation (30).

For the higher-order susceptibilities, asymptotic agreement at large frequencies between the results obtained using the method of moments and the high-frequency formulae (28) and (29) is obvious from figures 1(*a*)–(*f*). Since the method of moments yields the correct frequency dependence of the susceptibilities at high frequencies and is analytically exact in the zero-frequency limit, it can be concluded that it is also accurate in the intermediate frequency range. Qualitatively similar dependences of the susceptibilities for the first three harmonics on frequency and close agreement between the results obtained from the three methods was found for other choices of the parameters.

4. Application of the method of moments for calculating the amplification parameters of the system

In this section, we study the dependence of the spectral amplification factor $|\chi^{(n)}|^2$ on noise intensity for the first three harmonics. All the results are obtained for the driving frequency $\Omega = 0.1$. The system under study is a stochastic oscillator in a biased quartic potential with the Hamiltonian given by equation (1). It is easy to show that for bias fields $h > h_0 = 2/(3\sqrt{3}) \approx 0.385$, the system becomes monostable.

Figures 2(*a*)–(*c*) show the dependence of the spectral amplification factor at the first harmonic on D for three values of the bias $h = 0$, 0.3 and 0.6, respectively. The corresponding plots were obtained using the method of moments as well as the numerical calculation of w_1 from equation (30); good agreement between the two groups of data is clearly seen. An SR peak is observed on all three plots. Increasing the bias h leads to the shifting of the position of the SR peak to higher noise intensities and to its suppression. An unexpected conclusion drawn from figure 2(*c*) is that even for a bias field so large that the system becomes monostable, SR is still observed as a peak of $|\chi^{(1)}|^2$ at some optimal noise intensity.

Qualitatively similar behaviour is found on the spectral amplification on the second harmonic, $|\chi^{(2)}|^2$, versus noise intensity graphs shown in figure 2. It is known [14] that for symmetric systems, corresponding to zero bias, all the even-harmonic susceptibilities are zero, and therefore we present only the results obtained for the bias fields slightly less ($h = 0.3$, figure 3(*a*)) and higher ($h = 0.6$, figure 3(*b*)) than h_0. In both cases, SR is marked by a peak of the amplification factor $|\chi^{(2)}|^2$ that occurs at some D regardless of whether the system is bistable (figure 2(*a*)) or monostable (figure 2(*b*)).

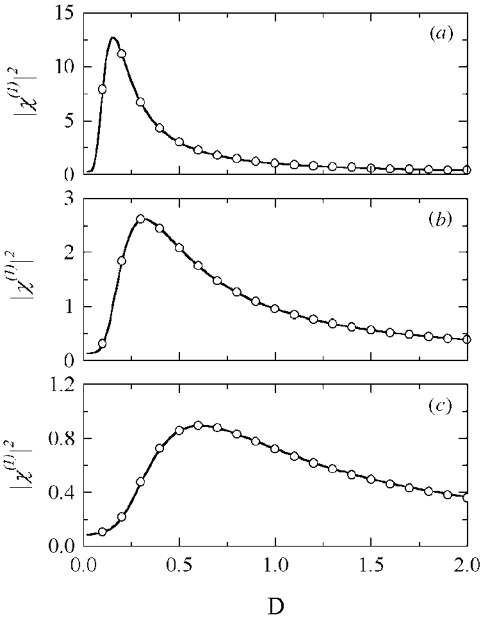

Figure 2. The dependence of the spectral amplification factor at the first harmonic on noise intensity. The values of the bias field are: (*a*) $h = 0$, (*b*) 0.3 and (*c*) 0.6. The graphs were calculated from the method of moments (full curves) and numerical solution of equation (30) (circles).

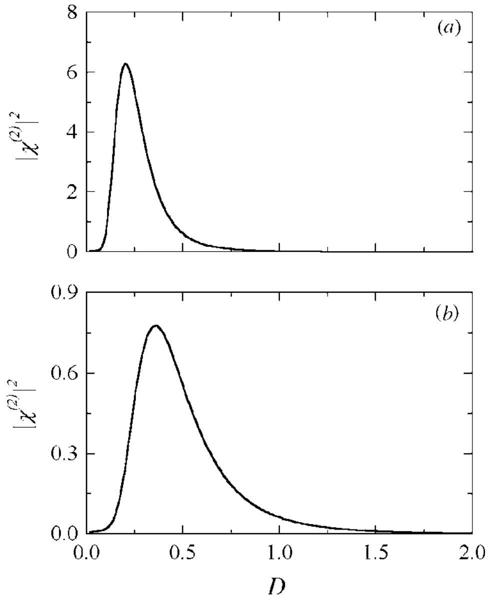

Figure 3. The dependence of the spectral amplification factor at the second harmonic on noise intensity. The values of the bias field are: (*a*) $h = 0.3$ and (*b*) 0.6.

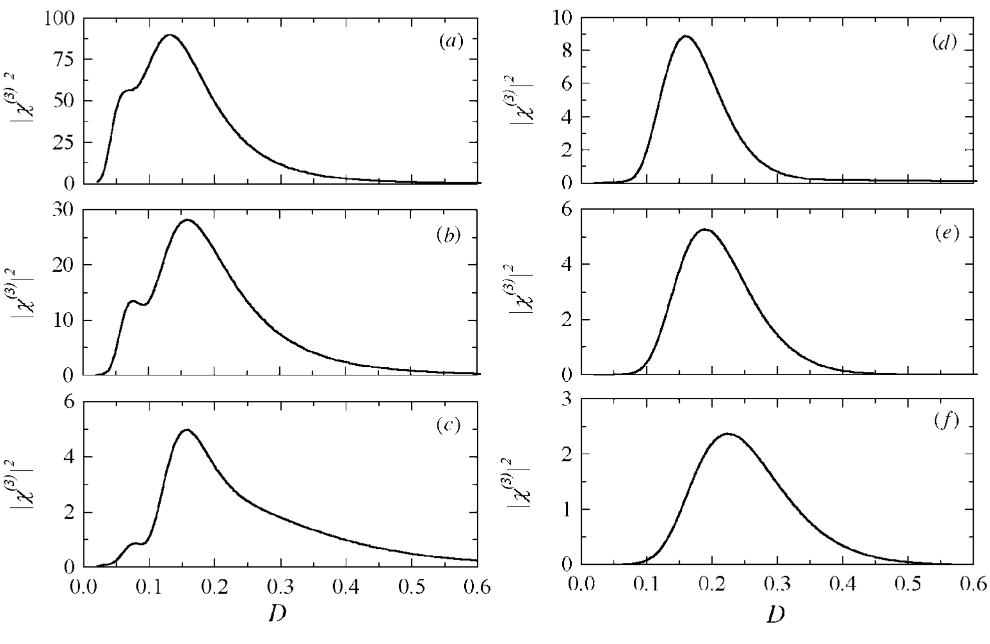

Figure 4. The dependence of the spectral amplification factor at the third harmonic on noise intensity. The values of the bias field are: (*a*) $h = 0$, (*b*) 0.1, (*c*) 0.2, (*d*) 0.3, (*e*) 0.4 and (*f*) 0.5.

The noise intensity dependence of the amplification factor for the third harmonic, $|\chi^{(3)}|^2$, is shown in figure 4. At zero bias (figure 4(*a*)), SR seems to occur *twice* at the third harmonic as there are two peaks on the $|\chi^{(3)}|^2$ versus D graphs. As the bias increases, the peaks become better resolved, but the low-D peak reduces in height (figure 4(*b*) and (*c*)) and disappears completely (figure 4(*d*)–(*f*)) as the bias increases further. Again, we note that even for large values of h that render the system monostable, SR still occurs on the third harmonic (figures 4(*d*)–(*f*)).

5. Discussion

The phenomenon of SR is usually observed in stochastic systems with more than one equilibrium state, such as a bistable overdamped oscillator. The physical explanation of SR that has been widely accepted (see, e.g., [19]) is that under the action of noise, the oscillator performs random flips from one equilibrium position to the other. Kramers [20] found that the average flipping frequency depends on the noise intensity in an Arrhenius-like manner

$$\nu_K \propto \mathrm{e}^{-\Delta U/D} \tag{34}$$

where ΔU is the height of the potential barrier separating the two attractors, and the proportionality constant is related to the curvature of the energy relief at the positions of stable and unstable equilibria. At some non-zero value of D, the Kramers frequency matches that of the driving field and thus the random flips of the oscillator become synchronized with external driving. This synchronization results in the enhancement of the system's sensitivity to external driving, and hence in the optimization of the signal-to-noise ratio and the spectral amplification factor.

Thus, common sense suggests that multistability of a potential is a necessary condition for SR, ruling out the possibility of SR in monostable systems. Nevertheless, it was proposed [21] that SR enhancement of the system's dynamic response has no relation to the matching of the driving frequency and the Kramers rate. Therefore, it is an interesting problem to find a monostable system that exhibits SR characteristics.

A non-conventional SR in monostable systems had been reported some time ago [22]. However, such a non-conventional SR was found only in *underdamped* systems and can be understood as the noise enhancement of the system's response under the condition of a matching of the driving frequency and the natural oscillation frequency of the system. Such intrawell SR was observed in bistable systems [23] as well. On the other hand, SR in an *overdamped* monostable system has been found recently [3]. However, the system studied in this case was a harmonic oscillator with only a weak nonlinearity that destroyed the symmetry of the potential but did not affect essentially the system's response at the first harmonic; thus the possibility of SR in such systems was established only for higher harmonics. It can be concluded from the presented results that, contrary to what can be expected, SR behaviour of the power amplification factor is possible in overdamped monostable systems not only at higher harmonics, but also at the first harmonic.

The method of moments allows for a simple interpretation of a SR peak at the first harmonic even in monostable nonlinear systems. Considering the equation of motion for the first moment only, i.e. truncating the power series (14) at $k = 1$, yields the following Debye expression for the first-order susceptibility:

$$\chi^{(1)} = \frac{\chi_{\mathrm{st}}^{(1)}}{1 + \mathrm{i}\Omega\tau}. \tag{35}$$

Here, the static susceptibility, $\chi_{st}^{(1)}$, is given by the first equation in the set of expressions (20), and the relaxation time, τ, in our dimensionless units is equal to $\chi_{st}^{(1)}$. Equation (35) yields qualitatively the same dependence of spectral amplification on frequency and noise intensity as that obtained from the inclusion of higher moments.

It is easy to verify that for zero bias, both $\chi_{st}^{(1)}$ and τ are monotonically decreasing functions of the noise intensity that tend to infinity at $D = 0$. Thus, the explanation of the SR peak at the first harmonic is based on the slowing down of the system at low noise intensities. Although the static susceptibility is large at low intensities of noise, the magnitude of the dynamic susceptibility (35) is small since the system cannot follow the temporal variations of the driving field in view of its large relaxation time. As the noise intensity increases, the relaxation time decreases, resulting in increasing the dynamic susceptibility to a maximum value at some optimal noise intensity. At still larger values of D, the relaxation time becomes small enough to allow for coherent oscillations of the average coordinate of the system with the driving field, and thus the dynamic susceptibility of the systems decreases with D as does its static susceptibility.

The case when the system is asymmetric, that is, biased potential, is easier to understand. It is not difficult to verify that for such a case $\chi_{st}^{(1)}$ itself has a maximum at some value of the noise intensity. Thus, the peak on the plot of the first-harmonic amplification factor versus D at $\Omega > 0$ is qualitatively similar to that observed on the $\left(\chi_{st}^{(1)}\right)^2$ versus D plot. The only effect of increasing the driving frequency is shifting of this peak to higher noise intensities and its suppression in height.

6. Conclusions

It is demonstrated that the method of moments is an efficient tool in the studies of the dynamic response of periodically driven stochastic systems. First-, second-, and third-order susceptibilities of a biased overdamped unharmonic stochastic oscillator are calculated from the method of moments and from the high-frequency asymptotic formulae derived in the present paper. The first-order susceptibility was also calculated numerically from the Fokker–Planck equation. Good agreement is found between the results obtained using these three methods.

The dependence of the spectral amplification of the first three harmonics on the noise intensity is investigated. It is shown that when the external bias becomes so large that the bistable system turns into a monostable one, SR-like behaviour is still observed as a peak of the spectral amplification factor on the first and higher harmonics versus noise intensity. Also, it is demonstrated that at not too large a bias, SR on the third harmonic is marked by two peaks of the power amplification factor as a function of noise intensity, as opposed to only one peak observed on the first and second harmonics.

Although the application of the method of moments was discussed in the context of dynamic response of an overdamped oscillator with only one coordinate, it is possible to extend this approach to a more general case of underdamped systems with several degrees of freedom. The analysis of this more general situation will be carried out in the future.

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